SHEDDING LIGHT ON THE SUBJECT

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in collaboration with

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Imagine yourself swimming in the ocean. Based on the photo in Figure 1, take a moment to make a conjecture for the graph of the light intensity as a function of your depth. Before continuing to read this article, select one of the graphs in Figure 2 which matches your conjecture. For most students, the fact that the light intensity decreases as the depth increases is intuitive. All of their conjectures illustrate this fact (Figure 2). However, students frequently disagree on the shape of the curve. These differences are a result of two informal yet incorrect observations. One belief is that if you go deep enough in the ocean, then there is no light and so the graph must reach zero. A second belief is that near the surface of the water, there is little change in the light intensity and so the curve must have an initial gradual change. This simple question of how the light intensity changes as a function of depth provides an excellent catalyst and center for the following light activity.

Fig. 1

An underwater snapshot is used to launch the activity.

In this activity, students develop exponential models, difference equations and procedures for analyzing exponential data. Students explore concepts and methods which have become fundamental in mathematics, chemistry, physics and biology. The rich context and hands-on experiments provide a strong foundation from which students can make sense of the principles and procedures involved. Some equipment needs to be gathered and prepared, but these materials are easily available. The experiments have the added features of being fast, simple, inexpensive and reliable which make their use in the classroom feasible.

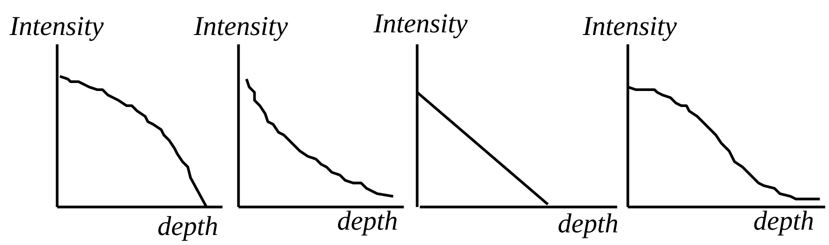


Fig. 2
These graphs are sample student-generated conjectures on the relationship between light intensity and depth.

After making conjectures about the light intensity, students gather data for light passing through layers of Plexiglas (see Figure 3) and through differing depths of water in a tube. Both experiments follow the same general format, but the Plexiglas data focuses on a discrete perspective of light passing through layers while the column of water promotes a continuous perspective. Thus both experiments may be performed in class with a discussion of the similarities and differences in the results, or just one experiment may be used depending on the objectives of the course. The difference between a discrete and continuous perspective can also be a launching point for calculus-based discussions.

Fig. 3

Students Collecting Data Through Layers of Plexiglas.

Once students have collected data and discussed the factors which might influence the change in light intensity, students recognize the need to produce three graphs: the light intensity versus depth, the change in light intensity versus depth, and the change in light intensity versus the light intensity (see Figure 4). From the graphs, students observe that the change in light intensity is linear with respect to the light intensity. Using this observation, students are in a position to develop what is known as *Beer-Lambert's Law*. By working at least initially from a discrete perspective, students have a solid mathematical footing to make sense of the key elements of this fundamental principle in science and mathematics.

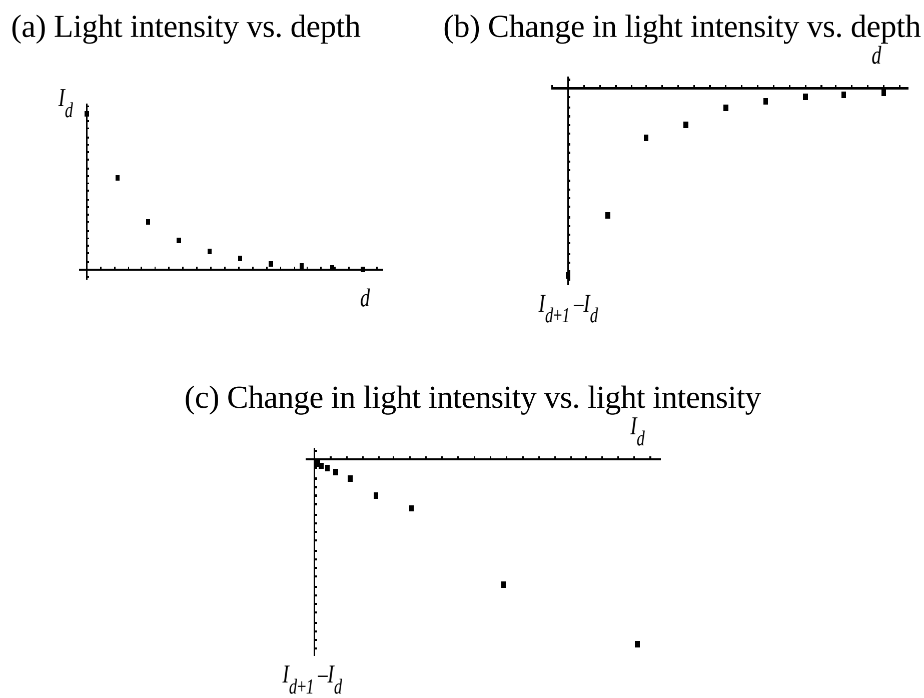


Fig. 4

Students produce three graphs to find a relationship they know. Students conjecture a linear relationship between the change in light intensity and the light intensity.

The development of Beer-Lambert's law is done from a phenomenological perspective (see Figure 5). First, students are asked to reason about the form of the linear relationship they observed. For example, students are asked to explain why the line must pass through the origin. If there is no light, then there can be no change in light. Thus, the observation that the change in the light intensity is linear with respect to the light intensity is equivalent to the difference equation:

$$I_{d+1} - I_d = k \cdot I_d$$

or that

$$I_{d+1} = (k + 1) \cdot I_d = c \cdot I_d$$

where I_d is the intensity of light at depth d, k is a constant between -1 and 0, and $c = k+1$. This statement is the basis of Lambert's law.

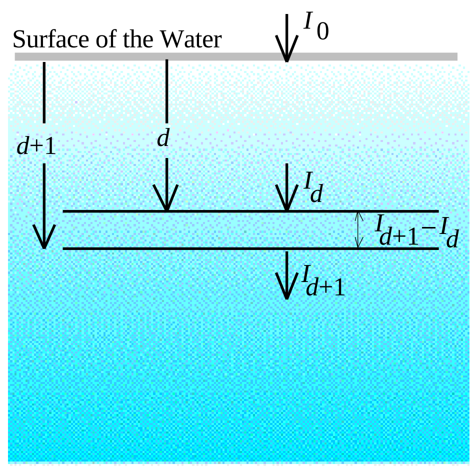


Fig. 5

A discrete model of light passes through 'layers' of water is used to develop Beer-Lambert's law.

Students can verify the reasonableness of the equation by observing that with each new depth, a fraction of the light is absorbed and that this fraction remains constant. Both statements resonate well with the Plexiglas experiment. Furthermore, students are asked to reason on the values of the constants. The constant k is negative for the fraction of light lost, while the constant c is positive representing the fraction of light retained.

Using these equations, students generate the pattern that $I_1 = c \cdot I_0$, $I_2 = c \cdot I_1 = c^2 \cdot I_0$, $I_3 = c^3 \cdot I_0$, ... from which the general solution of $I = c^d \cdot I_0$ can be conjectured and tested against the data. As already noted, this model of light absorbance is known historically as Lambert's law or more recently as *Beer-Lambert's law*:

$$I = I_0 \cdot e^{m \cdot d} = I_0 \cdot c^d$$

where I_o is the intensity of the incoming light and $m = \ln(c)$ is the absorption coefficient of the substance (Iavorskii, 1980). As a result of this activity, students are able to make sense of Beer-Lambert's law using their previous experiences, their experiences in conducting the experiments and symbolic reasoning.

Beer-Lambert's law can also be developed from a calculus perspective by considering increasingly small changes in depth. This results in a differential equation rather than a difference equation. An alternative definition of the absorption coefficient is as the constant of proportionality between the rate of change in the light intensity and the light intensity, $I' = m \cdot I$. Separation of variables or Euler's method can be used to solve the differential equation. In doing so, the definition of the natural logarithm and exponential functions as well as their derivatives and integrals arise naturally. Thus, this activity is useful in a wide-variety of classes including algebra, pre-calculus, calculus and differential equations.

The value of c in the discrete development is equal to the constant e^m. Sometimes the value of c is more meaningful since it more directly expresses the fraction of light absorbed. Discussion of how constants are defined can provide useful insight into the nature of science and mathematics and in this case, provide practice with exponents. Finally it should be noted that Beer-Lambert's law can also be developed from a statistical many-body perspective, but that requires much more mathematical sophistication.

By completing this investigation on Beer-Lambert's law, students have an opportunity to generate, collect, and analyze data. Students also have the opportunity to develop and test conjectures, recognize patterns, fit curves to make predictions established by the data, and represent the situation using recurrence relations. Beer-Lambert's law serves as the basis of spectroscopic instruments which are increasingly being used in the science curriculum. Moreover, research is still in progress to understand and find appropriate models for the light absorbance (e.g. Gordon, 1989; Perovich, 1995). The activity also leads to numerous deeper science and mathematics questions such as why the ocean appears blue and how does the light intensity change if only certain wavelengths are absorbed. For these reasons, the *Shedding Light on the Subject* has become one of our students' favorite activities.

Electronic versions of the activity pages are available at http://www.math.iastate.edu/keller. Other activities developing Lambert's law without the use of data driven experiments may be found in the **Journal of Chemical Education** articles “Discovering the Beer-Lambert Law” by Robert Ricci, Mauri A. Ditzler, and Lisa P. Nestor (1994), and “The Beer-Lambert Law Revisited: A Development without Calculus” by Peter Lykos (1992). These articles also provide a further discussion of the underlying physics as well as indicating numerous other teaching resources on Beer-Lambert’s law.

TEACHER NOTES

Prerequisites: Students should have some experience with the use and development of recurrence relations and comfort with either subscript or function notation. This activity can be done as an introduction to exponential functions or after students have studied basic exponential functions.

Grade levels: 11-12 Precalculus and Calculus

Materials: For Experiment 1, each student needs activity sheets and a graphics calculator, and each group needs access to windows or other light source (overhead projector or flashlight), CBL with light sensor, and 8-10 layers of tinted Plexiglas 1/8 to 1/4 inch wide available at most hardware stores and easily cut down to size. Four inch squares work nicely. If cut into larger 4” x 6” rectangles, then a Plexiglas rectangle can also be used as an inexpensive Mira which students can easily take home at night.

For Experiment 2, each student needs activity sheets and a graphics calculator, and each group needs a flashlight, CBL with light sensor, tube with false bottom, graduated cylinder or syringe to hold 30-50 mL of water, and lake water (tap water with food coloring will work). One way of making a tube with a false clear bottom is to use a golf club tube and a clear colorless 35mm film case. Insert the film case into the bottom of the tube about 5 inches (see Figure 6a). The two items have the same diameter and form a tight seal together. *Keep in mind that the light sensor is not water proof.* To hold the light sensor in place, pack foam rubber around it and place inside the tube.

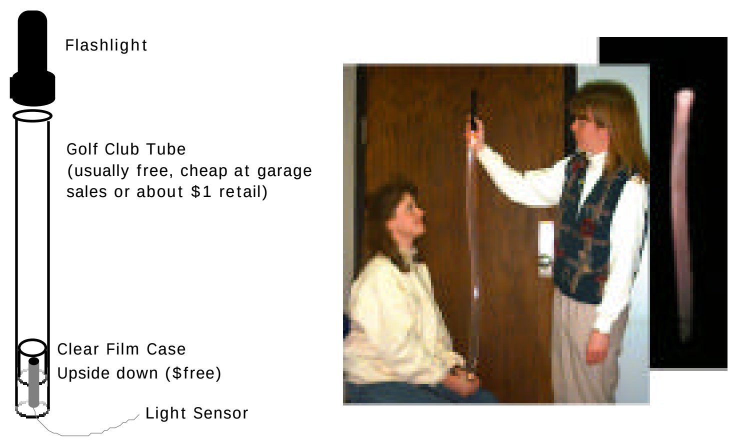


Fig. 6 (a) and (b)

Tubes for (a) collecting data or (b) demonstration can be produced with minimal cost.

Directions: The first activity sheet provides an introduction of the investigation. To assess the students' prior knowledge of the situation to be modeled, have the students complete Sheet 1 in their student groups. Additionally, the absorption of light can be demonstrated by stacking layers of Plexiglas on an overhead projector. Ask the students what they observe as each layer is added. Alternatively, a flashlight in a dark room through a clear tube filled with water provides an excellent visual aid (see Figure 6b).

Sheet 1: While students are completing the first question, make sure each student sketches a possible graph of the *light intensity* vs. *depth.* After most groups have finished discussing questions (a)-(e), have different groups sketch their graph for (a) on the board and explain their reasoning for the sketch. As a class, discuss the shapes of the graph. Students frequently conjecture that the graph is a line with negative slope or a parabola opening downward. Focus students' attention to what can be said about the end behavior of the light intensity and the vertical intercept. One method of increasing students' attention to details on their conjectures is to indicate that light intensity is often measured in lumens and ask students to label both axes with appropriate units and scale.

As a class, generate a list of items that influence the change in light intensity taking into account the different substances found in oceans. For (c), guide the students to the idea that the amount of light leaving a certain depth is dependent on the amount of light reaching that depth. This, in turn, leads to the discussion that the differences in the light intensity between two depths should be decreasing as the depth increases. Also, this discussion motivates why examining the relationship between the change in light intensity and the light intensity itself makes sense.

If possible, hold back distributing the remaining activity sheets. Give students a few minutes to discuss how they might create an experiment to measure the light intensity as it passes through various depths. This discussion will help to increase students' ownership of the experiments as well as making any comments on how to conduct the experiment more relevant.

Before students collect any data, demonstrate collecting readings using the overhead and Plexiglas. Initially, the sensor is likely to be over-powered and give false readings. This provides an opportunity to indicate data points at the extreme ranges, i.e. near 0.08 or above 0.9 may be questionable.

Sheets 2-3: For Experiment 1, the students should work in their groups to collect data using layers of tinted Plexiglas to represent the depths of water. The readings from the sensor may fluctuate widely after the addition of each layer of tinted Plexiglas. Students must choose a method for recording the light intensity readings. Algorithms that students have chosen include using the maximum, minimum, or average of the readings displayed on the CBL. Students should record the CBL readings of the light intensity and may enter them into their calculator after all the data is collected. *One strength of this activity is that the CBL does not have to be connected to the calculator*. No program needs to be downloaded and the equipment is easier to manage since it is not connected to a calculator. Since eight to ten readings are adequate, students can easily enter the data by-hand into their calculators. Students can be reminded of an appropriate range of values for the light readings. With the CBL and light sensor in its default settings, the best readings will fall between 0.008 and 0.91.

Direct sunlight or a high powered flashlight may over-power the light sensor resulting in inconsistent data. Thus, students may need additional layers of tinted Plexiglas to collect an adequate number of readings. If the data does seem to be unreasonable, you may encourage them to collect different readings using another light source or sunlight that is less intense.

After the data is collected, students may graph their data using graph paper or a graphics calculator checking that all the points seem reasonable and follow a curve. Students may then compute the difference in intensity between consecutive layers of Plexiglas. After computing the difference in intensities, students should plot the three graphs indicated. As the students explore the plots, they should realize that the plot $I(d+1)–I(d)$ vs. $I(d)$ gives a linear relationship. If not all the points for $I(d+1)–I(d)$ vs. $I(d)$ follow a line, encourage them to disregard those few points that are outliers. Thus an equation for the difference in intensities $I(d+1)–I(d)$ dependent on the light intensity may be found using linear regression on a graphing calculator or may be a point of review of algebra for the students.

The students should get an equation of the form $I(d+1) - I(d) = m \; I(d) + b$. The y-intercept should be very close to zero. Students should be asked to explain why the line should pass through the origin. Their responses can include that when light intensity is zero, the change in light intensity should be zero since there is no light to absorb. Thus, b should be disregarded leaving the equation $I(d+1) - I(d) = m \; I(d)$. Encourage the students to make sense of the equation by asking for the significance of the size and sign of the slope m. Since m is between (-1) and 0, the size of m indicates how much light is being absorbed, and the sign of m indicates that the difference in light intensity is decreasing. Thus, they should reason that the change in intensity of the light is a fraction of the intensity of the light entering the current layer of Plexiglas. Students can then solve their equation for $I(d+1)$ and find a recurrence relation $I(d+1) = (m+1)\ I(d)$ which can be used to approximate the light intensity at each layer of the tinted Plexiglas. Since $m+1$ is between 0 and 1, students should understand that the intensity of the light reaching the next depth is a fraction of the intensity entering the current depth. This recurrence equation may be plotted on graph paper or using the graphing calculator.

The recurrence relation can be used to generate an exponential function. The light intensity after the first layer can be written as $I(1) = (m+1)\ I(0)$. The light intensity at the second depth can expressed by $I(2) = (m+1)\ I(1) = (m+1)^2\ I(0)$. Students then can conjecture what the light intensity should be at layer *d*. Students can check to see how well their exponential function models their data by plotting the recurrence relation or exponential function on the same axes as the data. At this time, you may want to discuss Lambert's law and the similarities and differences between the equation the students found and the equation given in Lambert's law.

In a class discussion, have the students summarize the procedures which generated the exponential function emphasizing the difference in light intensity, the linear relationship between $I(d+1)–I(d)$ and $I(d)$, and the use of the recurrence relation to create the exponential function. Or, have the students write a short summary on the process explaining the methods used to generate the exponential function. Students need to include comments as to how well the exponential function fits the data and the reasons for any discrepancies.

Sheets 3–4: For Experiment 2, the procedures of this experiment follow in much the same manner as Experiment 1. In the student groups, one student records the data, another holds the tube steady, another fetches the water, and another handles the flashlight. In recording the data, students should maintain consistency by using their chosen algorithm to record the readings detected by the CBL light sensor. The analysis of the data is the same as that in Experiment 1. Unlike the Plexiglas experiment, the column of water experiment can use variable depths. Some students will want to explore this feature.

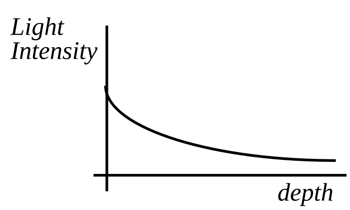
Extensions: Both Experiment 1 and Experiment 2 may be enhanced to be used in a calculus setting. Using that $I(d+1)–I(d) = \frac{I(d+1)–I(d)}{(d+1)–d}$ is the approximate rate of change, the students may replace $I(d+1)–I(d)$ with $DI(d)$ denoting an approximate to the derivative $I'(d)$. An additional follow-up question as part of the introduction with Sheet 1 is to ask students to indicate what the results of parts (d) and (e) infer about the rate of change in the light intensity $DI(d)$. At step (c) of Sheet 2 or 4, the concept of differential equations may be discussed as $I(d+1)–I(d) = m \quad I(d)$ becomes $DI(d) \approx m \quad I(d)$. After the students have generated an exponential function using the recurrence relation, they may be guided to see that based on the differential equation $DI(d) \approx m \quad I(d)$, the derivative of an exponential function is a constant multiplied by the original exponential function. Euler's Method may then be used to generate an approximation to the data based on the initial reading I_o and the differential equation $I'(d) = m \quad I(d)$. Alternatively, separation of variables can be used to solve the differential equation. Upon separating the variables, the integral of $I'(d)/I(d)$ is found to be the natural logarithm of I where I is the light intensity. A review of exponential and logarithmic functions leads to Lambert's law and the exponential decay function, $I(d) = I_o \; e^{m\,d}$ where m is the slope of the line found in part (c), $m<0$, and I_o is the initial reading of light intensity.

Assessment: Upon completion of these experiments, students may be asked to write an individual or group report describing what they learned and questions that have been generated. In writing a paper, students should formalize their understanding of the concepts and reflect on the way they came to understand the mathematics. Activities which encourage reflection allow students to analyze the development of their own mathematical ideas. Self-monitoring and evaluation of understanding are promoted. In addition, the instructor may use the papers to check each student's understanding of the material.

ANSWERS

Sheet 1

1. (a) The light intensity decreases and asymptotically approaches the horizontal axis as the depth increases.



(b) Answers can include the amount of substance in the water such as vegetation, sludge, waste, marine life, and the water's absorption of light.

(c) At 10 feet, 15 feet, and 20 feet, some of the factors influencing the light intensity are the amount of cloud cover, vegetation at the surface of the water, microscopic organisms, and the absorption of light by the water above. Most of the factors remain fairly constant as the depth changes.

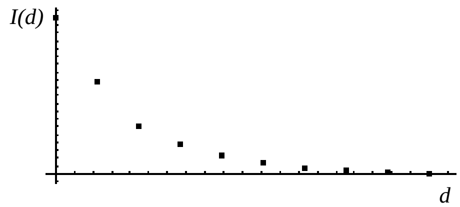
(d) The quantity $I(11)–I(10)$ is the change in light intensity between the depths of 10 feet and 11 feet.

(e) The quantity $I(12)–I(11)$ would be closer to zero than $I(11)–I(10)$ since the light intensity at 12 feet is just a fraction of the intensity at 11 feet which itself is a fraction of the light intensity at 10 feet. The difference is a function of the light intensity at the previous depth of water.

2.

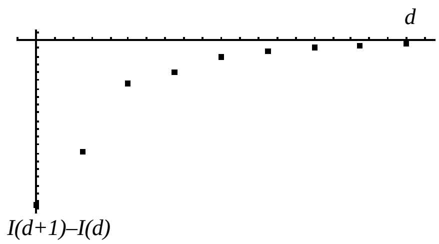
depth d	light intensity $I(d)$	$I(d+1)–I(d)$
0	0.810	-0.338
1	0.472	-0.230
2	0.242	-0.088
3	0.154	-0.065
4	0.089	-0.035
5	0.054	-0.023
6	0.031	-0.014
7	0.017	-0.009
8	0.008	

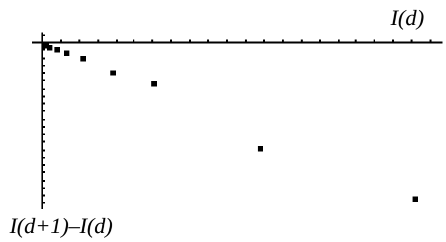
(a)



(b) See the table above.

(c)





The relationship between the change of intensity and light intensity is linear, which makes it easier to find an equation.

(d) $I(d+1)–I(d) = –0.43 \cdot I(d)+5.79\times10^{-5}$. The y-intercept is relatively close to zero due to the fact that when light intensity is zero, the difference in light intensity is close to zero. Thus it can be neglected. This leaves the equation $I(d+1) = 0.57 \cdot I(d)$. Since –0.43 is between –1 and 0, the size is the amount of light absorbed by the layers, and 0.57 is the amount of light remaining. The sign of –0.43 indicates that the difference in light intensities is decreasing.

(e) The initial value should be the first light intensity reading, $I(0) = I_o = 0.81$. Deviations are due to fluctuations in the light intensity readings, and real-world data.

(f) $I(1) = .57\ \ 0.81$, $I(2) = .57\ \ I(1) = .57^2\ \ 0.81$

$I(3) = .57\ \ I(2) = .57^3\ \ 0.81$, $I(d) = .57^d\ \ 0.81$

(g) (i) The light intensity $I(d)$ is a decreasing exponential function which depends on the number of layers of Plexiglas or the depth of the water.

(ii) Given a set of data, you could plot the data, examine the end behavior of the plot, and check to see if it appears to have a horizontal asymptote. If so, then you can plot the change in consecutive data points against the original data to determine if there seems to be a linear relationship. If there is a linear relationship and the line that represents that relationship goes through the origin, then a function similar to the light intensity data function would model the data. Note: If the calculus extensions are used, students may use separation of variables to verify that a linear relationship between the rate of change of the data and the data reveals that the original data may be modeled by an exponential function.

3. (a)-(g) similar to 2 (a)-(g) above.

(g) (iii) In Experiment 1 and Experiment 2 the methods of analyzing the data were the same since we examined the three plots and found a linear relationship between the change in light intensity, $I(d+1)–I(d)$ and the light intensity, $I(d)$. After finding the equation of a line fitting that relationship, we could express it as a recurrence relationship and develop an exponential function. The differences in the experiments included that Experiment 1 used discrete layers of Plexiglas while there is a continuum of water in the tube in Experiment 2. Readings with the column of water could be taken over different changes in depth.

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